Introduction

We used matrices in Chapter 2 simply to organize our work. It is time we examined them as interesting objects in their own right. There is much that we can do with matrices besides row operations: We can add, subtract, multiply, and even, in a sense, "divide" matrices. We use these operations to study game theory and input-output models in this chapter, and Markov chains in a later chapter.

Many calculators, electronic spreadsheets, and other computer programs can do these matrix operations, which is a big help in doing calculations. However, we need to know how these operations are defined to see why they are useful and to understand which to use in any particular application.

3.1 Matrix Addition and Scalar Multiplication

Let's start by formally defining what a matrix is and introducing some basic terms.

Matrix, Dimension, and Entries

An $m \times n$ **matrix** A is a rectangular array of real numbers with m rows and n columns. We refer to m and n as the **dimensions** of the matrix. The numbers that appear in the matrix are called its **entries.** We customarily use capital letters $A, B, C, \ldots$ for the names of matrices.

quick Examples

1. $A = \begin{bmatrix} 2 & 0 & 1 \\ 33 & -22 & 0 \end{bmatrix}$ is a 2×3 matrix because it has 2 rows and 3 columns.

2. $B = \begin{bmatrix} 2 & 3 \\ 10 & 44 \\ -1 & 3 \\ 8 & 3 \end{bmatrix}$ is a 4×2 matrix because it has 4 rows and 2 columns.

The entries of A are 2, 0, 1, 33, -22, and 0. The entries of B are the numbers 2, 3, 10, 44, -1, 3, 8, and 3.

Hint: Remember that the number of rows is given first and the number of columns second. An easy way to remember this is to think of the acronym "RC" for "Row then Column."

Referring to the Entries of a Matrix

There is a systematic way of referring to particular entries in a matrix. If i and j are numbers, then the entry in the ith row and jth column of the matrix A is called the $\boldsymbol{ij}$**th entry** of A. We usually write this entry as a_{ij} or A_{ij}. (If the matrix was called B, we would write its ijth entry as b_{ij} or B_{ij}.) Notice that this follows the "RC" convention: The row number is specified first and the column number second.

quick Example

With $A = \begin{bmatrix} 2 & 0 & 1 \\ 33 & -22 & 0 \end{bmatrix}$,

$a_{13} = 1$ First row, third column

$a_{21} = 33$ Second row, first column

According to the labeling convention, the entries of the matrix A above are

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

In general, the $m \times n$ matrix A has its entries labeled as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

We say that two matrices A and B are **equal** if they have the same dimensions and the corresponding entries are equal. Note that a 3×4 matrix can never equal a 3×5 matrix because they do not have the same dimensions.

using *Technology*

See the Technology Guides at the end of the chapter to see how matrices are entered and used in a TI-83/84 or Excel. For the authors' web-based utility, follow:

Chapter 3
→ Tools
→ Matrix Algebra Tool

There you will find a computational tool that allows you to do matrix algebra. Use the following format to enter the matrix *A* on the previous page (spaces are optional):

```
A = [2, 0, 1
33, -22, 0]
```

To display the matrix *A*, type `A` in the formula box and press "Compute."

Example 1 Matrix Equality

Let $A = \begin{bmatrix} 7 & 9 & x \\ 0 & -1 & y+1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 9 & 0 \\ 0 & -1 & 11 \end{bmatrix}$. Find the values of x and y such that $A = B$.

Solution For the two matrices to be equal, we must have corresponding entries equal, so

$$x = 0 \qquad a_{13} = b_{13}$$
$$y + 1 = 11 \quad \text{or} \quad y = 10 \qquad a_{23} = b_{23}$$

+ *Before we go on...* Note in Example 1 that the matrix equation

$$\begin{bmatrix} 7 & 9 & x \\ 0 & -1 & y+1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 0 & -1 & 11 \end{bmatrix}$$

is really six equations in one: $7 = 7, 9 = 9, x = 0, 0 = 0, -1 = -1$. and $y + 1 = 11$. We used only the two that were interesting. ■

Row Matrix, Column Matrix, and Square Matrix

A matrix with a single row is called a **row matrix,** or **row vector.** A matrix with a single column is called a **column matrix** or **column vector.** A matrix with the same number of rows as columns is called a **square matrix.**

***quick* Examples**

The 1×5 matrix $C = [3 \quad -4 \quad 0 \quad 1 \quad -11]$ is a row matrix.

The 4×1 matrix $D = \begin{bmatrix} 2 \\ 10 \\ -1 \\ 8 \end{bmatrix}$ is a column matrix.

The 3×3 matrix $E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 4 \\ -4 & 32 & 1 \end{bmatrix}$ is a square matrix.

Matrix Addition and Subtraction

The first matrix operations we discuss are matrix addition and subtraction. The rules for these operations are simple.

Matrix Addition and Subtraction

Two matrices can be added (or subtracted) if and only if they have the same dimensions. To add (or subtract) two matrices of the same dimensions, we add (or subtract) the corresponding entries. More formally, if A and B are $m \times n$ matrices, then $A + B$ and $A - B$ are the $m \times n$ matrices whose entries are given by:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$ ijth entry of the sum = sum of the ijth entries

$$(A-B)_{ij} = A_{ij} - B_{ij}$$ ijth entry of the difference = difference of the ijth entries

Visualizing Matrix Addition

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

quick Examples

1. $\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 9 & -5 \\ 0 & 13 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ 1 & 13 \\ -2 & 6 \end{bmatrix}$ Corresponding entries added

2. $\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 9 & -5 \\ 0 & 13 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 1 & -13 \\ 0 & 0 \end{bmatrix}$ Corresponding entries subtracted

Example 2 Sales

The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

using *Technology*

See the Technology Guides at the end of the chapter to see how to add and subtract matrices using a TI-83/84 or Excel. Alternatively, use the Matrix Algebra Tool at

Chapter 3
→ Tools
→ Matrix Algebra Tool

There, first enter the two matrices you wish to add or subtract (subtract, in this case) as shown:

```
J = [20, 15, 10
12, 8, 4]

F = [23, 12, 8
12, 4, 5]
```

To compute their difference, type `F-J` in the formula box and press "Compute." (You can enter multiple formulas separated by commas in the formula box. For instance, `F+J, F-J` will compute both the sum and difference.)

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

Solution The tables suggest two matrices:

$$J = \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix}$$

To compute the change in sales of each product for both stores, we want to subtract corresponding entries in these two matrices. In other words, we want to compute the difference of the two matrices:

$$F - J = \begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 0 \\ -4 & 1 \end{bmatrix}$$

Thus, the change in sales of each product is the following:

	Vancouver	**Quebec**
Wiper Blades	3	−3
Cleaning Fluid (bottles)	−2	0
Floor Mats	−4	1

Scalar Multiplication

A matrix A can be added to itself because the expression $A + A$ is the sum of two matrices that have the same dimensions. When we compute $A + A$, we end up doubling every entry in A. So we can think of the expression $2A$ as telling us to *multiply every element in* A *by* 2.

In general, to multiply a matrix by a number, multiply every entry in the matrix by that number. For example,

$$6\begin{bmatrix} \frac{5}{2} & -3 \\ 1 & 0 \\ -1 & \frac{5}{6} \end{bmatrix} = \begin{bmatrix} 15 & -18 \\ 6 & 0 \\ -6 & 5 \end{bmatrix}$$

It is traditional when talking about matrices to call individual numbers **scalars.** For this reason, we call the operation of multiplying a matrix by a number **scalar multiplication.**

Example 3 Sales

The revenue generated by sales in the Vancouver and Quebec branches of the A-Plus auto parts store (see Example 2) was as follows:

January Sales in Canadian Dollars

	Vancouver	**Quebec**
Wiper Blades	140.00	105.00
Cleaning Fluid	30.00	36.00
Floor Mats	96.00	48.00

using *Technology*

See the Technology Guides at the end of the chapter to see how to compute scalar multiples using a TI-83/84 or Excel. Alternatively, go to the online Matrix Algebra Tool at

Chapter 3
→ Tools
→ Matrix Algebra Tool

There, enter the January sales in U.S. Dollars:

```
A = [140, 105
30, 36
96, 48]
```

Then type `0.65*A` in the formula box and press "Compute."

If the Canadian dollar was worth \$0.65 U.S. at the time, compute the revenue in U.S. dollars.

Solution We need to multiply each revenue figure by 0.65. Let A be the matrix of revenue figures in Canadian dollars:

$$A = \begin{bmatrix} 140.00 & 105.00 \\ 30.00 & 36.00 \\ 96.00 & 48.00 \end{bmatrix}$$

The revenue figures in U.S. dollars are then given by the scalar multiple

$$0.65A = 0.65 \begin{bmatrix} 140.00 & 105.00 \\ 30.00 & 36.00 \\ 96.00 & 48.00 \end{bmatrix} = \begin{bmatrix} 91.00 & 68.25 \\ 19.50 & 23.40 \\ 62.40 & 31.20 \end{bmatrix}$$

In other words, in U.S. dollars, \$91 worth of wiper blades was sold in Vancouver, \$68.25 worth of wiper blades was sold in Quebec, and so on.

Formally, scalar multiplication is defined as follows:

Scalar Multiplication

If A is an $m \times n$ matrix and c is a real number, then cA is the $m \times n$ matrix obtained by multiplying all the entries of A by c. (We usually use lowercase letters c, d, e, ... to denote scalars.) Thus, the ijth entry of cA is given by

$$(cA)_{ij} = c(A_{ij})$$

In words, this rule is: To get the ijth entry of cA, multiply the ijth entry of A by c.

Example 4 Combining Operations

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ 5 & -6 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} x & y & w \\ z & t+1 & 3 \end{bmatrix}$

Evaluate the following: $4A$, xB, and $A + 3C$.

Solution First, we find $4A$ by multiplying each entry of A by 4:

$$4A = 4\begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ 12 & 20 & -12 \end{bmatrix}$$

Similarly, we find xB by multiplying each entry of B by x:

$$xB = x\begin{bmatrix} 1 & 3 & -1 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} x & 3x & -x \\ 5x & -6x & 0 \end{bmatrix}$$

We get $A + 3C$ in two steps as follows:

$$A + 3C = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + 3\begin{bmatrix} x & y & w \\ z & t+1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 3x & 3y & 3w \\ 3z & 3t+3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3x & -1+3y & 3w \\ 3+3z & 3t+8 & 6 \end{bmatrix}$$

Addition and scalar multiplication of matrices have nice properties, reminiscent of the properties of addition and multiplication of real numbers. Before we state them, we need to introduce some more notation.

If A is any matrix, then $-A$ is the matrix $(-1)A$. In other words, $-A$ is A multiplied by the scalar -1. This amounts to changing the signs of all the entries in A. For example,

$$-\begin{bmatrix} 4 & -2 & 0 \\ 6 & 10 & -6 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ -6 & -10 & 6 \end{bmatrix}$$

For any two matrices A and B, $A - B$ is the same as $A + (-B)$. (Why?)

Also, a **zero matrix** is a matrix all of whose entries are zero. Thus, for example, the 2×3 zero matrix is

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we state the most important properties of the operations that we have been talking about:

Properties of Matrix Addition and Scalar Multiplication

If A, B, and C are any $m \times n$ matrices and if O is the zero $m \times n$ matrix, then the following hold:

$A + (B + C) = (A + B) + C$	*Associative law*
$A + B = B + A$	*Commutative law*
$A + O = O + A = A$	*Additive identity law*
$A + (-A) = O = (-A) + A$	*Additive inverse law*
$c(A + B) = cA + cB$	*Distributive law*
$(c + d)A = cA + dA$	*Distributive law*
$1A = A$	*Scalar unit*
$0A = O$	*Scalar zero*

These properties would be obvious if we were talking about addition and multiplication of *numbers,* but here we are talking about addition and multiplication of *matrices*. We are using "+" to mean something new: matrix addition. There is no reason why matrix addition has to obey *all* the properties of addition of numbers. It happens that it does obey many of them, which is why it is convenient to call it *addition* in the first place. This means that we can manipulate equations involving matrices in much the same way that we manipulate equations involving numbers. One word of caution: We haven't yet discussed how to multiply matrices, and it probably isn't what you think. It will turn out that multiplication of matrices does *not* obey all the same properties as multiplication of numbers.

Transposition

We mention one more operation on matrices:

Transposition

If A is an $m \times n$ matrix, then its **transpose** is the $n \times m$ matrix obtained by writing its rows as columns, so that the ith row of the original matrix becomes the ith column of the transpose. We denote the transpose of the matrix A by A^T.

Visualizing Transposition

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 \\ -3 & 0 & 1 \end{bmatrix}$$

quick Examples

1. Let $B = \begin{bmatrix} 2 & 3 \\ 10 & 44 \\ -1 & 3 \\ 8 & 3 \end{bmatrix}$. Then $B^T = \begin{bmatrix} 2 & 10 & -1 & 8 \\ 3 & 44 & 3 & 3 \end{bmatrix}$.

4 × 2 matrix   2 × 4 matrix

2. $[\,-1 \quad 1 \quad 2\,]^T = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

1 × 3 matrix   3 × 1 matrix

using *Technology*

See the Technology Guides at the end of the chapter to see how to transpose a matrix using a TI-83/84 or Excel. Alternatively, go to the online Matrix Algebra Tool at

Chapter 3
→ Tools
→ Matrix Algebra Tool

There, first enter the matrix you wish to transpose:

```
A = [2, 0, 1
33, -22, 0]
```

Then type `A^T` in the formula box and press "Compute."

Properties of Transposition

If A and B are $m \times n$ matrices, then the following hold:

$$\begin{aligned} (A + B)^T &= A^T + B^T \\ (cA)^T &= c(A^T) \\ (A^T)^T &= A \end{aligned}$$

To see why the laws of transposition are true, let us consider the first one: $(A + B)^T = A^T + B^T$. The left-hand side is the transpose of $A + B$, and so is obtained by first adding A and B, and then writing the rows as columns. This is the same as first writing the rows of A and B individually as columns before adding, which gives the right-hand side. Similar arguments can be used to establish the other laws of transposition.